Section 7.3: Properties of the Laplace Transform

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Translation in s

Theorem 3. If the Laplace transform $\mathscr{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then

(1) $$\mathscr{L}\{e^{at}f(t)\}(s) = F(s-a)$$

for $s > \alpha + a$.

What does this mean?

Section 7.3: Properties of the Laplace Transform

Translation in s

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(1) $\mathscr{L}\{e^{at}f(t)\}(s) = F(s-a)$

for $s > \alpha + a$.

Proof:

Section 7.3: Properties of the Laplace Transform

Example 1 Determine the Laplace transform of $e^{at} \sin bt$.

Section 7.3: Properties of the Laplace Transform

Laplace Transform of the Derivative

Theorem 4. Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then, for $s > \alpha$,

(2) $$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0)\ .$$

What does this mean?

Section 7.3: Properties of the Laplace Transform

Laplace Transform of the Derivative

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Proof:

Section 7.3: Properties of the Laplace Transform

Laplace Transform of Higher-Order Derivatives

Theorem 5. Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

(4) $$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) .$$

Illustrate for $n = 2$

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Laplace Transform of Higher-Order Derivatives

Theorem 5. Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\{f^{(n)}\}(s) = s^n\mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) . \quad (4)$$

Proof for $n = 2$:

Section 7.3: Properties of the Laplace Transform

Laplace Transform of Higher-Order Derivatives

Theorem 5. Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\{f^{(n)}\}(s) = s^n \mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \, . \tag{4}$$

Illustrate for general n

Section 7.3: Properties of the Laplace Transform

Example 2 Using Theorem 4 and the fact that $\mathcal{L}\{\sin bt\}(s) = \dfrac{b}{s^2 + b^2}$, determine $\mathcal{L}\{\cos bt\}$.

Section 7.3: Properties of the Laplace Transform

Example 3 Prove the following identity for continuous functions $f(t)$ (assuming the transforms exist):

(5) $$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s) .$$

Use it to verify the solution to Example 2.

Section 7.3: Properties of the Laplace Transform

Derivatives of the Laplace Transform

Theorem 6. Let $F(s) = \mathcal{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then, for $s > \alpha$,

(6) $$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s).$$

What does this mean?

Section 7.3: Properties of the Laplace Transform

Derivatives of the Laplace Transform

Theorem 6. Let $F(s) = \mathcal{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then, for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)\,. \tag{6}$$

Proof:

Section 7.3: Properties of the Laplace Transform

Example 4 Determine $\mathscr{L}\{t \sin bt\}$.

Section 7.3: Properties of the Laplace Transform

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2 + b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$

TABLE 7.2 Properties of Laplace Transforms

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\} .$$

$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\} \qquad \text{for any constant } c .$$

$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a) .$$

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0) .$$

$$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0) .$$

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) .$$

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right) .$$